1. Complexes

1.1. We consider sets endowed with an order relation denoted by $\subset$ and read "is a face of" or "is contained in" . Such a set is called a complex if the ordered subset of all faces of any given element is isomorphic with the ordered set of all subsets of a set, and if any two elements A,B have a greatest lower bound, denoted by $A \cap B$. A complex has a smallest element which we shall always denote by 0 . The number of minimal non 0 faces of an element A is called the rank of A , and denoted by $\operatorname{rk} A$. The elements of rank 1 are called vertices. Since an element of a complex is completely characterized by the set of its vertices, we may also define a complex as a set Δ of subsets of a set V (the set of vertices), such that $\{x\} \in \Delta$ for all $x \in V$, and that $B \subset A \in \Delta$ implies $B \in \Delta$; the rank of an element of Δ is its cardinality (as a subset of V) . The rank of a complex Δ , denoted by $\operatorname{rk} \Delta$, is by definition $\sup \{\operatorname{rk} A \mid A \in \Delta\}$. A complex is called a simplex if it is isomorphic to the set of all subsets of a given set, ordered by inclusion. Hereafter, Δ always denotes a complex.

We define a morphism $\alpha : \Delta \to \Delta'$ of Δ into another complex Δ' as a mapping of the underlying sets such that, for every $A \in \Delta$, the restriction of α to the simplex of all faces of A is an isomorphism of ordered sets onto the simplex of all faces of $\alpha(A)$. (About later use of the word 'morphism' in these notes, see also the general convention at the end of n° 1.3) .

A subcomplex of Δ is by definition a complex Δ' whose underlying set is a subset of Δ , and such that the inclusion is a morphism (this means that the order relation of Δ' is induced by that of Δ , and that if $A \in \Delta'$, all faces of A in Δ belong to Δ') .

If $A \in \Delta$, the set of all elements of Δ which contain A , together with the order relation induced by that of Δ , is a complex, called the star of A

(in Δ) and denoted by St A .(Notice the slight deviation from the terminology commonly used in topology, where St A is often called the link of A in Δ .) If $B \in$ St A , the rank of B in St A is called the codimension of A in B , and denoted by $\operatorname{codim}_B A$.

If two elements $A, B \in \Delta$ have an upper bound, they are called incident (another deviation from the standard terminology!); in that case, they have a least upper bound, denoted by $A \cup B$.

A sequence $A_0, A_1, \ldots, A_m$ of elements of a complex is called a chain if, for every $i = 0, 1, \ldots, m-1$, one of the two relations $A_i \subset A_{i+1}$ or $A_{i+1} \subset A_i$ holds.

If Δ and Δ' are two complexes, the direct product of their underlying sets, ordered by

$$(A,A') \subset (B,B') \quad \text{iff} \quad A \subset A' \quad \text{and} \quad B \subset B' \quad ,$$

is a complex $\Delta * \Delta'$, called the join of the two complexes.

1.2. Let V be a set endowed with a reflexive and symmetric relation, called the incidence. The subsets of V whose points are pairwise incident, ordered by inclusion, form a complex F(V) whose vertices are the sets reduced to one point. A complex is called a flag complex if it is isomorphic to such an F(V) .

Let Δ , Δ' be two complexes and let V , V' be their sets of vertices. A morphism $\Delta \to \Delta'$ is completely characterized by its restriction to V . A necessary condition for a mapping $V \to V'$ to be the restriction of a morphism $\Delta \to \Delta'$ is that it maps pairs of distinct incident vertices onto pairs of distinct incident vertices; if Δ' is a flag complex, this condition is also sufficient.

1.2.1. PROPOSITION. *A complex Δ is a flag complex if and only if every set of pairwise incident elements of Δ has an upper bound. If Δ has finite rank, it is a flag complex if and only if every triplet of pairwise incident elements has an upper bound.*

The proof is immediate.

1.3. A complex Δ is called a *chamber complex* if every element is contained in a maximal element and if, given two maximal elements C, C', there exists a finite sequence

$$C = C_0 , C_1 , \ldots , C_m = C' \tag{1}$$

such that

$$\operatorname{codim}_{C_{i-1}}(C_{i-1} \cap C_i) = \operatorname{codim}_{C_i}(C_{i-1} \cap C_i) \leqslant 1 \tag{2}$$

for all $i = 1, \ldots, m$. The maximal elements of Δ are then called *chambers*. From now on, *the letter Δ will always denote a chamber complex*. The set of all chambers contained in a subset L of a chamber complex is denoted by $\mathrm{Cham}_\Delta L$, or simply by $\mathrm{Cham}\, L$ when no confusion can arise.

An element $A \in \Delta$ has the same codimension in all chambers containing it. Indeed, let $C, C' \in \mathrm{Cham}\,\Delta$ be such that $A \subset C \cap C'$, let $(C_i)_{i=0, \ldots, m}$ be as above, and let $B = C_0 \cap C_1 \cap \ldots \cap C_m \cap A$. Then, it follows immediately from (2), by induction on i, that $\operatorname{codim}_{C_i} B = \operatorname{codim}_C B$, so that $\operatorname{codim}_C B = \operatorname{codim}_{C'} B$, and, in view of the fact that $\operatorname{codim}_A B < \infty$, we can write

$$\operatorname{codim}_C A = \operatorname{codim}_C B - \operatorname{codim}_A B =$$

$$= \operatorname{codim}_{C'} B - \operatorname{codim}_A B = \operatorname{codim}_{C'} A \ .$$

The common value of all $\operatorname{codim}_C A$, for $C \in \operatorname{Cham} \Delta$ and $C \supset A$ is called the codimension of A (in Δ) and denoted by $\operatorname{codim} A$.

If $C, C' \in \operatorname{Cham} \Delta$, all elements of a sequence (1) satisfying (2) are chambers, as is immediately deduced from (2), by induction on i and using what we have just said of the codimension. Such a sequence is called a gallery of length m and extremities C, C' (or joining C and C'). For $A, A' \in \Delta$, the gallery (1) is said to be stretched from A to A' (or between A and A') if $A \subset C$, $A' \subset C'$ and if there is no gallery of strictly smaller length with the same properties; the length m is then called the distance of A and A' , and denoted by $\operatorname{dist} AA'$. The gallery (1) is called minimal if it is stretched between its extremities, that is if $m = \operatorname{dist} CC'$. By definition, a gallery (1) stammers if there is an $i \in \{0, 1, \ldots, m-1\}$ such that $C_{i+1} = C_i$; in particular, a minimal gallery cannot stammer. The diameter of Δ , denoted by $\operatorname{diam} \Delta$, is defined as the $\sup \{\operatorname{dist} CC' \mid C, C' \in \operatorname{Cham} \Delta\}$. The fundamental property of chamber complexes, that any two chambers can be joined by a gallery, will be referred to as the connectedness property.

Two chambers C, C' are called adjacent if $\operatorname{dist} CC' = 1$, that is, if $\operatorname{codim} (C \cap C') = 1$. Thus a finite sequence of chambers is a gallery if and only if any two consecutive elements of the sequence are identical or adjacent.

A chamber complex is called thick (resp. thin) if every element of codimension 1 is contained in at least three (resp. exactly two) chambers.

A morphism of a chamber complex into another is called a morphism of chamber complexes if it maps the chambers onto chambers. Such a morphism obviously maps the galleries onto galleries, and diminishes the distance of chambers. In these notes, we shall deal almost exclusively with chamber complexes. Except when otherwise specified, all morphisms which we shall consider will be morphisms of chamber complexes.

1.4. Let G be a group, let I be a set, for every $i \in I$ let G^i be a subgroup of G, and for every subset $\underline{\underline{i}} \subset I$ set

$$G_{\underline{\underline{i}}} = \bigcap_{i \notin \underline{\underline{i}}} G^i .$$

In the set $\Gamma = \coprod_{\underline{\underline{i}} \subset I} G/G_{\underline{\underline{i}}}$, we introduce the following order relation: for $A \in G/G_{\underline{\underline{i}}}$ and $B \in G/G_{\underline{\underline{j}}}$, we set $A \subset B$ iff $\underline{\underline{i}} \supset \underline{\underline{j}}$ and if A, viewed as a subset of G (a coset of $G_{\underline{\underline{i}}}$), *contains* the subset B in the set-theoretical sense; if all $G_{\underline{\underline{i}}}$ are different, this simply means that we provide Γ with the order relation which is the opposite of the set-theoretical inclusion. Together with this relation, Γ is a complex which will be denoted by $\underline{C}(G;(G_{\underline{\underline{i}}})_{\underline{\underline{i}} \subset I})$, and on which G operates by left translations. The star of the element $G_{\underline{\underline{j}}}$ $(\underline{\underline{j}} \subset I)$ in $\underline{C}(G;(G_{\underline{\underline{i}}}))$ clearly is the complex $\underline{C}(G_{\underline{\underline{j}}};(G_{\underline{\underline{i}}})_{\underline{\underline{i}} \subset \underline{\underline{j}}})$; in particular, if Γ is a chamber complex, the number of chambers containing $G_{\underline{\underline{j}}}$ is the index $[G_{\underline{\underline{j}}} : G_{\emptyset}]$, and the complex is thin iff $[G_{\{i\}} : G_{\emptyset}] = 2$ for all $i \in I$.

It is an easy exercise to show that, for three subgroups X, Y, Z of a group, the following three properties are equivalent

(1) $(X.Y) \cap (X.Z) = X.(Y \cap Z)$;

(2) $(X \cap Y).(X \cap Z) = X \cap (Y.Z)$;

(3) If three cosets xX, yY, zZ have pairwise a non-empty intersection, then $xX \cap yY \cap zZ \neq \emptyset$.

1.4.1. PROPOSITION. *Let* I, G, $(G_{\underline{\underline{i}}})_{\underline{\underline{i}} \subset I}$ *be as above. Then, if* I *is finite, the complex* $\underline{C}(G;(G_{\underline{\underline{i}}}))$ *is a flag complex iff any three subgroups* $G_{\underline{\underline{i}}}$, $G_{\underline{\underline{j}}}$, $G_{\underline{\underline{k}}}$ ($\underline{\underline{i}}$, $\underline{\underline{j}}$, $\underline{\underline{k}}$ *subsets of* I) *possess the equivalent properties* (1), (2) *and* (3). *In any case,* $\underline{C}(G;(G_{\underline{\underline{i}}}))$ *is a chamber complex iff the subgroups* $G_{\{i\}}$ $(i \in I)$ *generate* G .

The first assertion follows from 1.2.1. The second is immediate.

1.5. In a chamber complex Δ , a set L of chambers is called convex if every minimal gallery whose extremities belong to L has all its terms in L . A chamber subcomplex Δ' of Δ (i.e. a subcomplex which is a chamber complex and such that the inclusion is a chamber morphism) is called convex if Cham Δ is convex. It is readily seen that if a chamber subcomplex is an intersection of convex chamber subcomplexes, it is itself convex. An arbitrary subcomplex of Δ is called convex if it is an intersection of convex chamber subcomplexes.

The full convex hull of a subset L of Δ is defined as the smallest convex subcomplex Δ' containing L . The set Cham Δ' is then called the convex hull of L . It is easily seen that the convex hull of a set of chambers is the smallest convex set of chambers containing it.

1.6. PROPOSITION. *Let Δ , Δ' be two chamber complexes such that every element of codimension 1 is contained in at most two chambers, let φ and ψ be two (chamber) morphisms of Δ into Δ' , and let $A, B \in$ Cham Δ . Assume that $\varphi \mid_{\text{Cham}\,\Delta}$ is injective and that φ and ψ coincide on the set of all faces of A . If φ and ψ do not coincide on the set of all faces of B , then dist $(\varphi(A)$, $\psi(B)) <$ dist AB , and $\psi \mid_{\text{Cham}\,\Delta}$ is not injective. Assume further that Δ is thin. Then, Δ' is also thin, φ is surjective and $\varphi \mid_{\text{Cham}\,\Delta}$ is distance preserving.*

Suppose that φ and ψ do not coincide on all faces of B , let $G = (A = A_0, A_1, \ldots, A_m = B)$ be a minimal gallery, and let i be the smallest integer such that φ and ψ do not coincide on all faces of A_i . Then, $\psi(A_i) \supset \psi(A_{i-1} \cap A_i) = \varphi(A_{i-1} \cap A_i)$. Therefore, $\psi(A_i) = \varphi(A_{i-1})$ or $\varphi(A_i)$. But the second possibility would clearly imply that φ and ψ coincide on all faces of A_i (because they already coincide on all faces of $A_{i-1} \cap A_i$). Therefore,

$\psi(A_i) = \varphi(A_{i-1}) = \psi(A_{i-1})$, and the gallery $\psi(G)$ stammers, which proves the first part of the proposition.

We now assume that Δ is thin and first show that

(1) If C' , $D' \in \text{Cham}\,\Delta'$ are adjacent, and if $C' = \varphi(C)$ with $C \in \text{Cham}\,\Delta$, then $\varphi^{-1}(D') = \{D\}$, where D is a chamber adjacent to C .

Indeed, let E be the face of codimension 1 of C such that $\varphi(E) = = C' \cap D'$, and let D be the chamber containing E and different from C . Then $\varphi(D)$ contains $C' \cap D'$ and is different from C' , because of the injectivity of φ , so that $\varphi(D) = D'$.

From (1) and the connectedness of Δ' follows immediately that $\varphi(\text{Cham}\,\Delta) = \text{Cham}\,\Delta'$; the surjectivity of φ ensues.

Two chambers $C, D \in \text{Cham}\,\Delta$ are adjacent if and only if $\varphi(C)$ and $\varphi(D)$ are adjacent. The "only if" is clear; the "if" is consequence of (1). Now, the fact that φ preserves the distance of chambers and the thinness of Δ follows readily.

1.7. COROLLARY. <u>If an endomorphism of a thin chamber complex is injective on the set of chambers and leaves invariant all faces of a given chamber, it is the identity.</u>

1.8. An endomorphism $\varphi : \Delta \to \Delta$ of a thin chamber complex is called a <u>folding</u> if it is idempotent and if every chamber C belonging to $\varphi(\Delta)$ is the image of exactly two chambers of Δ by φ . One of these two chambers is necessarily

C itself; we denote the other by $\bar{\varphi}(C)$. If C is a chamber not belonging to $\varphi(\Delta)$, we set $\bar{\varphi}(C) = C$.

1.9. LEMMA. *Let $\varphi : \Delta \to \Delta$ be a folding. Then, the images by $\bar{\varphi}$ of two adjacent chambers are identical or adjacent.*

Let C , D be the two chambers. We may assume that at least one of them, say C , belongs to $\varphi(\Delta)$. Let A be the face of $\bar{\varphi}(C)$ whose image by φ is $C \cap D$, and let D' be the chamber which contains A and is different from $\bar{\varphi}(C)$. Then, $\varphi(D')$ contains $C \cap D$, so that it coincides with C or D . If $D' \notin \varphi(\Delta)$, we must have $\varphi(D') = D$ because $D' \not\prec \bar{\varphi}(C)$, so that $D' = \bar{\varphi}(D)$. If $D' \in \varphi(\Delta)$, then also $A \in \varphi(\Delta)$ so that $A = \varphi(A) = C \cap D$, and $\bar{\varphi}(C) = D = \bar{\varphi}(D)$.

1.10. PROPOSITION. *Let $\varphi : \Delta \to \Delta$ be a folding.*

(i) *There exists a pair of adjacent chambers such that one of them belongs to $\varphi(\Delta)$ and the other not; if $\{C,C'\}$ is such a pair, with $C \in \varphi(\Delta)$, then $\varphi(C') = C$.*

(ii) *Both the set $\varphi(\text{Cham}\,\Delta)$ and its complement in $\text{Cham}\,\Delta$ are convex.*

(iii) *If C , C' are as in (i) and if D is any chamber,* $\text{dist}\, C'D - \text{dist}\, CD = 1$ *or* -1 *according as D belongs to $\varphi(\Delta)$ or not.*

(iv) *If C , C' are as in (i), then φ is the only folding which maps C' onto C .*

We first show that

(1) If C , C' are two adjacent chambers and if $C \in \varphi(\Delta)$ and $C' \notin \varphi(\Delta)$, then $\varphi(C') = C$ and $\bar{\varphi}(C) = C'$.

Indeed, since $C \cap C' \in \varphi(\Delta)$, it is invariant by φ so that $\varphi(C')$ contains $C \cap C'$ and must coincide with C .

Let now G be any minimal gallery having elements both in $\varphi(\Delta)$ and in its complement. Such a gallery clearly possesses two consecutive elements C , C' such that one of them, say C , belongs to $\varphi(\Delta)$ while the other does not. This already proves (i). From (1), it follows that $\varphi(C') = C$ and $\bar{\varphi}(C) = C'$, so that the galleries $\varphi(G)$ and $\bar{\varphi}(G)$ stammer ($\bar{\varphi}(G)$ is a gallery in view of lemma 1.9); consequently, the extremities of G cannot both belong to $\varphi(\Delta)$ (resp. to $\complement\varphi(\Delta)$) otherwise the stammering gallery $\varphi(G)$ (resp. $\bar{\varphi}(G)$) would have the same extremities as G , which is impossible. This establishes (ii).

Let C , C' , D be as in (iii) and assume first that $D \in \varphi(\Delta)$. Take for G a minimal gallery joining D and C' . Then, the stammering gallery $\varphi(G)$ joins D and C , and one has

$$\text{dist } DC \leqslant \text{dist } DC' - 1 \ .$$

But the opposite inequality also holds, obviously, so that we have equality. If $D \notin \varphi(\Delta)$, one proves in a similar way, taking now for G a gallery joining D and C and using $\bar{\varphi}(G)$ instead of $\varphi(G)$, that $\text{dist } DC' = \text{dist } DC - 1$.

There remains to prove (iv). Let ψ be any folding such that $\psi(C') = C$ and let us denote by $\bar{\varphi}(\Delta)$ the smallest subcomplex of Δ containing the set $\bar{\varphi}(\text{Cham } \Delta)$; this set being convex, by (ii), $\bar{\varphi}(\Delta)$ is a chamber complex. From (iii) it follows that $\psi(\text{Cham } \Delta) = \varphi(\text{Cham } \Delta)$ and that $\bar{\psi}(\text{Cham } \Delta) = \bar{\varphi}(\text{Cham } \Delta)$. Applying the proposition 1.6 to the restrictions of φ and ψ to $\varphi(\Delta)$ and the chamber C, we see that $\varphi|_{\varphi(\Delta)} = \psi|_{\varphi(\Delta)}$. Similarly $\varphi|_{\bar\varphi(\Delta)} = \psi|_{\bar\varphi(\Delta)}$. Consequently, $\varphi = \psi$.

1.11. COROLLARY. Let $\varphi : \Delta \to \Delta$ be a folding and let C, C' be two adjacent chambers such that $\varphi(C') = C$. Assume that there exists a folding φ' such that $\varphi'(C) = C'$. Then $\varphi(\Delta) \cup \varphi'(\Delta) = \Delta$, the intersection $\varphi(\Delta) \cap \varphi'(\Delta)$ does not contain any chamber, and φ' is the only folding having these two properties. The mappings φ' and $\bar\varphi$ coincide on Cham Δ. There exists an involutory automorphism $\rho : \Delta \to \Delta$ which coincides with φ on $\varphi'(\Delta)$ and with φ' on $\varphi(\Delta)$.

The relations $\varphi(\Delta) \cup \varphi'(\Delta) = \Delta$ and Cham $(\varphi(\Delta) \cap \varphi'(\Delta)) = \emptyset$ are immediate consequences of 1.10 (iii). If φ'' is any folding satisfying these two relations, one has $C' \in \varphi''(\Delta)$, $C \notin \varphi''(\Delta)$, thus $\varphi''(C) = C'$ by 1.10 (i), and $\varphi'' = \varphi'$ by 1.10 (iv).

If an element of Δ belongs to $\varphi(\Delta) \cap \varphi'(\Delta)$, it is invariant by both φ and φ'; from the relation $\varphi(\Delta) \cup \varphi'(\Delta) = \Delta$, it then follows that the morphisms $\varphi'|_{\varphi(\Delta)}$ and $\varphi|_{\varphi'(\Delta)}$ can be glued into an endomorphism $\rho : \Delta \to \Delta$. This endomorphism is injective on Cham Δ, and its square leaves invariant each face of C, therefore (1.7) ρ^2 is the identity and ρ is an automorphism. Finally, if $D \in$ Cham $\varphi(\Delta)$, one has $\varphi(\varphi'(D)) = \rho(\rho(D)) = D$, so that $\varphi'(D) = \bar\varphi(D)$, and the same is clearly true for $D \in$ Cham $\varphi'(\Delta)$.

1.12. When it exists, the folding φ' of corollary 1.11 will be called opposite to φ and denoted by $\bar\varphi$ (which is consistent with the notation introduced in 1.8). The automorphism ρ is then called the reflection associated with φ. Let $B \in \Delta$ have codimension 1; if there is a reflection ρ leaving B invariant, this reflection is unique (because if $B = C \cap C'$, with C, $C' \in$ Cham Δ, one has

$\rho(C) = C'$, and ρ must be the reflection associated with a folding mapping C onto C'), and is called the reflection with respect to B .

The image $\varphi(\Delta)$ of a folding is called a root, or the root associated with φ . The wall of a root Φ , denoted by $\partial\Phi$, is by definition the subcomplex of Δ consisting of all $A \in \Delta$ such that there exist adjacent chambers $C \in \Phi$ and $C' \notin \Phi$ with $A \subset C \cap C'$. Two roots Φ , $\bar{\Phi}$ are called opposite if they are associated with opposite foldings; by 1.11, this means that $\Phi \cup \bar{\Phi} = \Delta$ and $\Phi \cap \bar{\Phi} \cap \text{Cham}\, \Delta = \emptyset$.

1.13. LEMMA. *Let C , C' be two adjacent chambers of the thin complex Δ , set $A = C \cap C'$ and let Γ be the set of all galleries G which are stretched between A and a chamber of Δ . Let φ and φ' be two endomorphisms of Δ leaving all faces of A invariant and such that $\varphi(C) = \varphi(C') = C$ and $\varphi'(C') = \varphi'(C) = C'$. Then, if φ and φ' map Γ in itself, they are opposite foldings.*

Let $G = (C_0, C_1, \ldots, C_m) \in \Gamma$ and set $C_m = D$. We first show by induction on $m = \text{dist}\, AD$ that

(1) If $C_0 = C$ (resp. C') , all faces of D are invariant by φ and $\varphi \circ \varphi'$ (resp. by φ' and $\varphi' \circ \varphi$);

(2) If $C_0 = C$ (resp. C') , then $\varphi'(D) \neq D$ (resp. $\varphi(D) \neq D$) .

For $m = 0$, both statements are obvious. Let m be > 0 . Without loss of generality, we may assume that $C_0 = C$. Set $\psi = \varphi$ or $\varphi \circ \varphi'$. By the induction hypothesis, ψ leaves invariant all faces of C_{m-1} . The chamber $\psi(D)$ contains $\psi(C_{m-1} \cap D) = C_{m-1} \cap D$, so that $\psi(D) = C_{m-1}$ or D , but the first possibility is excluded by the assumption made on φ and φ' . Therefore, $\psi(D) = D$. Since ψ

also leaves invariant all faces of $C_{m-1} \cap D$, it leaves invariant all faces of D , and (1) is proved. Now suppose that $\varphi'(D) = D$. Applying (1) to the gallery $\varphi'(G)$, for which it is already proved, we see that $\varphi'(C_{m-1} \cap D) = C_{m-1} \cap D$, from which follows that $\varphi'(C_{m-1}) = C_{m-1}$, which contradicts the induction hypothesis. Therefore, $\varphi'(D) \neq D$.

Let now D be any chamber of Δ , and let G be an element of Γ whose last term is D . Applying (1) to $\varphi(G)$, we see that φ leaves invariant all faces of $\varphi(D)$, so that $\varphi^2(B) = \varphi(B)$ for all $B \subset D$, and the morphism φ is idempotent.

From (1) and (2), it follows that for every chamber E , either $\varphi(E) = \varphi(\varphi'(E)) = E \neq \varphi'(E)$ or $\varphi'(E) = \varphi'(\varphi(E)) = E \neq \varphi(E)$. Now, let $D \in \text{Cham}\, \varphi(\Delta)$, and let $E \in \varphi^{-1}(D)$. Then, either $E = \varphi(E) = D$, or $E = \varphi'(\varphi(E)) = \varphi'(D)$. In other words, $\varphi^{-1}(D) = \{D, \varphi'(D)\}$. Since $\varphi'(D) \neq D$, this shows that φ is a folding. The same holds for φ' , and the lemma is proved.